

Analytic representations in the complex plane and unit disc

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Euclidean case: complex plane

- Background formalism
- Gaussian states
- Bargmann representation
(growth)
- zeros of Bargmann functions
(growth/density of zeros)
- Hadamard's theorem

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Background formalism

- Hilbert space: $f(x)$ where $x \in R$

$$(f, g) = \int_{-\infty}^{\infty} dx [f(x)]^* g(x)$$

x -representation

normalize $(f, f) = 1$

- orthonormal basis: 'number states'

$$u_N(x) = \pi^{-1/4} [2^N N!]^{-1/2} H_N(x) \exp\left(-\frac{1}{2}x^2\right)$$

$$f(x) = \sum_N f_N u_N(x); \quad f_N = \int f(x) u_N(x) dx$$

$N=0,1,2,\dots$

- p -representation

$$\tilde{f}(p) = (2\pi)^{-1/2} \int (dx) f(x) e^{-ixp}$$

- position/momentum operators:

$$x - \text{repr} : \quad x; \quad p = -i\partial_x$$

$$p - \text{repr} : \quad x = i\partial_p; \quad p$$

$$[x, p] = i1$$

also:

$$\begin{aligned}
 a &= 2^{-1/2}(x + ip) = 2^{-1/2}(x + \partial_x) \\
 a^\dagger &= 2^{-1/2}(x - ip) = 2^{-1/2}(x - \partial_x) \\
 [a, a^\dagger] &= 1
 \end{aligned}$$

$$\begin{aligned}
 au_N(x) &= \sqrt{N}u_{N-1}(x) \\
 a^\dagger u_N(x) &= \sqrt{N+1}u_{N+1}(x) \\
 a^\dagger au_N(x) &= Nu_N(x)
 \end{aligned}$$

- Phase space: $x - p$ plane
function $f(x)$ located around $(\langle x \rangle, \langle p \rangle)$
uncertainty ellipse $\Delta x, \Delta p, \sigma_{xp}$:

$$\begin{aligned}
 \langle x^M \rangle &= \int dx x^M |f(x)|^2; & \langle p^M \rangle &= \int dp p^M |\tilde{f}(p)|^2 \\
 \Delta x &= [\langle x^2 \rangle - \langle x \rangle^2]^{1/2}; & \Delta p & \\
 \sigma_{xp} &= \langle \frac{1}{2}(xp + px) \rangle - \langle x \rangle \langle p \rangle
 \end{aligned}$$

- displacements in phase space:

$$\begin{aligned}
 D(z) &= \exp[za^\dagger - z^*a] \\
 D(z)f(x) &= f(x - \sqrt{2}z_R) \exp(i\sqrt{2}z_I x - iz_R z_I)
 \end{aligned}$$

average location of function $f(x)$ displaced:

$$(\langle x \rangle, \langle p \rangle) \rightarrow (\langle x \rangle + \sqrt{2}z_R, \langle p \rangle + \sqrt{2}z_I)$$

Heisenberg-Weyl group

Gaussian states

- For normalized Gaussians

$$(\Delta x \Delta p)^2 - \sigma_{xp}^2 = \frac{1}{4}$$

- **coherent states :**
Gaussians with $\Delta x = \Delta p = 2^{-1/2}$; $\sigma_{xp} = 0$

'vacuum' (centered at (0,0))

$$u_0(x) = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2\right)$$

displace it

$$g(x; z) = D(z)u_0(x) = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + xz\sqrt{2} - zz_R\right)$$

centered at $(z_R\sqrt{2}, z_I\sqrt{2})$

- **squeezed states : other Gaussians**

- $g(x; z)$ for all $z \in C$: overcomplete basis
resolution of the identity

$$\int_C d\mu(z) g(x; z) [g(y; z)]^* = \delta(x - y); \quad d\mu(z) = \frac{dz_R dz_I}{\pi}$$

arbitrary $f(x)$

$$f(x) = \int_C d\mu(z) g(x; z) F(z); \quad F(z) = (g(x; z), f(x))$$

- non-orthogonal

$$(g(x; z), g(x; w)) = \exp\left(-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + z^* w\right)$$

- eigenstates of a :

$$ag(x; z) = zg(x; z)$$

- 'temporal stability'

For a class of Hamiltonians, they evolve into other coherent states within the family

$$\begin{aligned} H &= \omega [a^\dagger a - za^\dagger - z^* a + |z|^2] \\ \exp(iHt)g(x; \zeta) &= \exp[i|z|^2 \sin \omega t] \\ &\times g(x; \zeta e^{i\omega t} + z(1 - e^{i\omega t})) \end{aligned}$$

Bargmann representation

- arbitrary function

$$f(x) = \sum_N f_N u_N(x)$$

represented with the Bargmann function

$$f_B(z) = \exp\left[\frac{1}{2}|z|^2\right] ([f(x)]^*, g(x; z)) = \sum_{N=0}^{\infty} f_N z^N (N!)^{-1/2}$$

analytic function of z in C .

- growth of entire function $f(z)$:

$$|f| \approx \exp(\sigma R^\rho); \quad R \rightarrow \infty$$

order ρ and type σ

$(\rho, \sigma) < (\rho', \sigma')$ if $\rho < \rho'$ or $\rho = \rho'$ and $\sigma < \sigma'$.

- scalar product

$$(f, g) = \int_C [f_B(z)]^* g_B(z) \exp(-|z|^2) d\mu(z)$$

growth of Bargmann functions smaller than $(\rho = 2, \sigma = 1/2)$.

- In Bargmann representation

$$u_N(x) \rightarrow \frac{z^N}{\sqrt{N!}}$$

$$a \rightarrow \partial_z; \quad a^\dagger \rightarrow z$$

$$au_N(x) = \sqrt{N}u_{N-1}(x); \quad a^\dagger u_N(x) = \sqrt{N+1}u_{N+1}(x)$$

- coherent states: order $\rho = 1$, $\sigma = |w|$:

$$g(x; w) \rightarrow g_B(z; w) = \exp \left[wz - \frac{1}{2}|w|^2 \right]$$

special case vacuum

$$g(x; 0) = u_0(x) \rightarrow g_B(z; 0) = 1$$

- squeezed states: order $\rho = 2$, $\sigma = |\alpha|/2$

$$f_B(z) = (1 - |\alpha|^2)^{1/4} \exp \left[\frac{1}{2}\alpha z^2 + \beta z + \gamma \right]; \quad |\alpha| < 1$$

- example with a given order ρ and σ :

$$f(x) = \mathcal{N} \sum_{N=0}^{\infty} f_N u_N(x); \quad f_N = \frac{e^{i\theta_N} \sigma^{N/\rho} (N!)^{1/2}}{\Gamma(\frac{N}{\rho} + 1)}$$

$$\mathcal{N} = \left[\sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{[\Gamma(\frac{N}{\rho} + 1)]^2} \right]^{-\frac{1}{2}}$$

$\{\theta_N\}$ are arbitrary phases.

\mathcal{N} finite, growth less than ($\rho = 2, \sigma = 1/2$).

- Operator Θ (in the $u_N(x)$ basis Θ_{MN}):

$$\mathcal{K}(z, \zeta^*; \Theta) = \sum_{M,N} \Theta_{MN} z^M (\zeta^*)^N$$

$$(\Theta f)(z) = \int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z, \zeta^*; \Theta) f(\zeta)$$

examples

$$\mathcal{K}(z, \zeta^*; \mathbf{1}) = \exp(z\zeta^*)$$

$$\mathcal{K}(z, \zeta^*; a) = \zeta^* \exp(z\zeta^*)$$

$$\mathcal{K}(z, \zeta^*; a^\dagger) = z \exp(z\zeta^*)$$

$\mathcal{K}(z, \zeta^*; \mathbf{1})$ reproducing kernel,

$$\int_C d\mu(\zeta) e^{-|\zeta|^2 + z\zeta^*} f(\zeta) = f(z)$$

a, a^\dagger differential and integral reprs:

$$\int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z, \zeta^*; a) f(\zeta) = \partial_z f(z)$$

$$\int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z, \zeta^*; a^\dagger) f(\zeta) = z f(z)$$

zeros of Bargmann functions

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$$f_B(z) = \exp \left[\frac{1}{2} |z|^2 \right] ([f(x)]^*, g(x; z)) = 0$$

$[f(x)]^*$ orthogonal to $g(x; z)$

- zeros of analytic functions isolated
if sequence $\{\zeta_N\}$ converges to (finite) w then
corresponding coherent states $g(x; \zeta_N)$ over-
complete set.

- Assume zeros $\zeta_1, \zeta_2, \zeta_3 \dots$ are such that

$$0 < |\zeta_1| \leq |\zeta_2| \leq |\zeta_3| \leq \dots; \quad \lim_{N \rightarrow \infty} |\zeta_N| = \infty$$

density of zeros (η, δ)

$$n(R) \approx \delta R^\eta; \quad R \rightarrow \infty$$

η convergence exponent: infimum of λ

$$\sum_{N=1}^{\infty} |\zeta_N|^{-\lambda} < \infty$$

$(\eta, \delta) < (\eta', \delta')$ if $\eta < \eta'$ or $\eta = \eta'; \delta < \delta'$.

- growth (ρ, σ) of analytic functions related to the density (η, δ) of their zeros

$$\eta \leq \rho; \quad \sigma \rho \leq \delta$$

- if sequence $\{\zeta_N\}$ has density **smaller/greater** than $(\eta = 2, \delta = 1)$ then corresponding coherent states $g(x; \zeta_N)$ **undercomplete/overcomplete** set.

- **example:**

$$\zeta_N = \left(\frac{N}{\delta_0} \right)^{1/\eta_0} e^{i\theta_N}$$

θ_N arbitrary phases

has density $\eta = \eta_0$ and $\delta = \delta_0$

$\{g(x, \zeta_N)\}$ undercomplete for $\eta_0 = 1.9$ and overcomplete for $\eta_0 = 2.1$

Hadamard's theorem

- Bargmann functions with given zeros $\{\zeta_N\}$
many with same zeros

$$f_B(z) = z^m \prod_{N=1}^{\infty} E(\zeta_N, p) \exp[Q_q(z)]$$

m multiplicity of zero at origin

$Q_q(z)$ is a polynomial of degree q

E are Weierstrass factors:

$$E(\zeta, 0) = \left(1 - \frac{z}{\zeta}\right)$$

$$E(\zeta, p) = \left(1 - \frac{z}{\zeta}\right) \exp \left[\frac{z}{\zeta} + \frac{z^2}{2\zeta^2} + \dots + \frac{z^p}{p\zeta^p} \right]$$

maximum of (p, q) (genus) does not exceed ρ

For Bargmann $p, q = 0, 1, 2$.

- $\exp[Q_0(z)]$ (constant) is vacuum
 $\exp[Q_1(z)]$ is coherent state
 $\exp[Q_2(z)]$ is squeezed state

$$z^m \rightarrow a^{\dagger m}$$

$$\hat{E}(\zeta_N, 0) = 1 - \frac{a^\dagger}{\zeta_N}$$

$$\hat{E}(\zeta_N, 1) = \left[1 - \frac{a^\dagger}{\zeta_N} \right] \exp \left[\frac{a^\dagger}{\zeta_N} \right]$$

$$\hat{E}(\zeta_N, 2) = \left[1 - \frac{a^\dagger}{\zeta_N} \right] \exp \left[\frac{a^\dagger}{\zeta_N} + \frac{a^{\dagger 2}}{2\zeta_N^2} \right]$$

- example: superposition of two Gaussians

$$f(x) = \mathcal{N}[g(x; i) - g(x; -i)] \rightarrow f_B(z) = 2i\mathcal{N}e^{-1/2} \sin z$$

Hadamard factorization:

$$\sin z = z \prod_{N=1}^{\infty} E(\zeta_N, 0) = z \prod_{N=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 N^2}\right)$$

$$\zeta_{2N-1} = -N\pi; \quad \zeta_{2N} = N\pi$$

all $g(x; N\pi)$ orthogonal to $f(x)$.

- **von Neumann lattice**

$\{z_{MN} = S^{1/2}(M + iN)\}$ where M, N integers and S area of lattice cell.

density is ($\eta = 2, \delta = S/\pi$)

corresponding $\{g(x; z_{MN})\}$ overcomplete when $S < \pi$; undercomplete when $S > \pi$.

- construct $f(x)$ orthogonal to all $\{g(x; z_{MN})\}$ in a von Neumann lattice with $S = 4$.

Weierstrass sigma function:

$$\sigma(z|1, i) = z \prod E(\zeta_{MN}, 2); \quad \zeta_{MN} = 2(M + iN)$$

where $(M, N) \neq (0, 0)$ and

$$E(\zeta_{MN}, 2) = \left[1 - \frac{z}{\zeta_{MN}}\right] \exp\left[\frac{z}{\zeta_{MN}} + \frac{z^2}{2\zeta_{MN}^2}\right]$$

answer: $f_B(z) = \mathcal{N}\sigma(z|1, i)$

Hyperbolic case: unit disc

1. Bergman space

- Background formalism
- $SU(1, 1)$ transformations
- Operators
- $SU(1, 1)$ coherent states

2. Hardy space

- Background formalism
- Inner and outer states
- Zeros of $f(z)$: Blaschke products

Bergman space: background formalism

- Bergman space:
analytic functions $g(z)$ in D :

$$\frac{1 + \alpha}{\pi} \int_D |g(z)|^p (1 - |z|^2)^\alpha dz_R dz_I = 1$$

where $-1 < \alpha$ and $p > 0$.

Here $p = 2$ and $\alpha = 2k - 2$ with $k = 1/2, 1, 3/2, \dots$
for $k = 1/2$, $\alpha = -1$, as limit

- Hilbert space H_k of $f(z; k)$:

$$\begin{aligned} (f(z; k), g(z; k)) &= \frac{2k - 1}{\pi} \int_D [f(z; k)]^* g(z; k) \\ &\quad \times (1 - |z|^2)^{2k} d\mu(z) \\ d\mu(z) &= \frac{dz_R dz_I}{(1 - |z|^2)^2} \end{aligned}$$

- orthonormal basis: ' $SU(1, 1)$ number states

$$u_N(z; k) = d(N; k) z^N; \quad d(N; k) = \left[\frac{\Gamma(N + 2k)}{\Gamma(N + 1)\Gamma(2k)} \right]^{1/2}$$

where $N = 0, 1, 2, \dots$

$$f(z; k) = \sum_N f_N d(N; k) z^N; \quad \sum_N |f_N|^2 = 1$$

$SU(1, 1)$ transformations

- The operators

$$K_+ = z^2 \partial_z + 2kz, \quad K_0 = z \partial_z + k, \quad K_- = \partial_z$$

generators of $SU(1, 1)$ group:

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0$$

Casimir:

$$K^2 = K_0^2 - \frac{1}{2} (K_+ K_- + K_- K_+) = k(k - 1) \mathbf{1}$$

Here $k = 1/2, 1, 3/2, \dots$ discrete series of $SU(1, 1)$ representations.

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$$K^2 u_N(z; k) = k(k - 1) u_N(z; k),$$

$$K_0 u_N(z; k) = (k + N) u_N(z; k),$$

$$K_- u_N(z; k) = [N(N + 2k - 1)]^{1/2} u_{N-1}(z; k)$$

$$K_+ u_N(z; k) = [(N + 1)(N + 2k)]^{1/2} u_{N+1}(z; k),$$

- $SU(1, 1)$ transformations

$$S(r, \theta, \lambda; k) = \exp \left[-\frac{1}{2} r e^{-i\theta} K_+ + \frac{1}{2} r e^{i\theta} K_- \right] \exp(i\lambda K_0)$$

$$r \geq 0; \quad 0 \leq \theta < 2\pi; \quad 0 \leq \lambda < 2\pi$$

also

$$S(z; k) = S(r, \theta, 0; k); \quad z = -e^{-i\theta} \tanh \frac{r}{2}$$

$$S(z_1; k) S(z_2; k) = S(w; k) \exp(-i\phi K_0)$$

$$w = \frac{z_1 + z_2}{1 + z_1^* z_2}; \quad \phi = 2 \arg(1 + z_1^* z_2)$$

- the transformations

$$f(z; k) \rightarrow S(r, \theta, \lambda; k) f(z; k)$$

implemented with Mobius trans:

$$f(z; k) \rightarrow f \left(\frac{az + b}{b^* z + a^*}; k \right) (b^* z + a^*)^{-2k}$$

where

$$a = e^{i\lambda/2} \cosh \frac{r}{2}, \quad b = e^{i\theta} \sinh \frac{r}{2}$$

Operators

- operators Θ (in the $u_N(z; k)$ basis Θ_{MN}):

$$\mathcal{K}(z, \zeta^*; \Theta) = \sum_{M, N=0}^{\infty} d(M; k) d(N; k) \Theta_{MN} z^M \zeta^{*N}$$

$$(\Theta f)(z; k) = \frac{2k-1}{\pi} \int_D \mathcal{K}(z, \zeta^*; \Theta) f(\zeta; k) (1 - |\zeta|^2)^{2k} d\mu(\zeta)$$

Example:

$$\mathcal{K}(z, \zeta^*; \mathbf{1}) = (1 - z\zeta^*)^{-2k}$$

reproducing kernel,

$$\frac{2k-1}{\pi} \int_D \mathcal{K}(z, \zeta^*; \mathbf{1}) f(\zeta; k) (1 - |\zeta|^2)^{2k} d\mu(\zeta) = f(z; k)$$

operators K_0, K_+, K_-

$$\mathcal{K}(z, \zeta^*; K_0) = \frac{2k[-(z\zeta^*)^2 + z\zeta^* + 1]}{(1 - z\zeta^*)^{2k}}$$

$$\mathcal{K}(z, \zeta^*; K_+) = \frac{2kz}{(1 - z\zeta^*)^{2k-1}}$$

$$\mathcal{K}(z, \zeta^*; K_-) = \frac{2k\zeta^*}{(1 - z\zeta^*)^{2k-1}}$$

consistent with differential representations

$SU(1, 1)$ coherent states

- $SU(1, 1)$ coherent states

$$\begin{aligned}
 h(z; w; k) &= S(z; k)u_0(z; k) \\
 &= (1 - |w|^2)^k \sum_{N=0}^{\infty} d(N; k)w^N u_N(z; k) \\
 &= \frac{(1 - |w|^2)^k}{(1 - zw)^{2k}}; \quad |z| < 1; \quad |w| < 1
 \end{aligned}$$

- resolution of the identity

$$\frac{2k - 1}{\pi} \int_D h(z; w; k) [h(\zeta; w; k)]^* d\mu(w) = (1 - z\zeta^*)^{-2k}$$

Arbitrary state $f(z; k)$

$$\begin{aligned}
 f(z; k) &= \frac{2k - 1}{\pi} \int_D h(z; w; k) F(w) d\mu(w) \\
 F(w) &= \frac{2k - 1}{\pi} \int_D f(\zeta; k) [h(\zeta; w; k)]^* (1 - |\zeta|^2)^{2k} d\mu(\zeta)
 \end{aligned}$$

- non-orthogonal states

$$(h(z; w_1; k), h(z; w_2; k)) = \frac{(1 - |w_1|^2)^k (1 - |w_2|^2)^k}{(1 - w_1^* w_2)^{2k}}$$

- for any state $f(z; k)$

$$f(z; k) = (1 - |z|^2)^{-k} (h(\zeta; z^*; k), f(\zeta; k))$$

$h(z; w^*; k)$ orthogonal to $f(z; k) \leftrightarrow w$ zero of $f(z; k)$

- zeros of analytic functions isolated
if sequence $\{w_N\}$ converges in the unit disc
then corresponding coherent states $h(z; w_N^*, k)$
overcomplete set.
- work needed to connect density of zeros in the
unit disc to growth near the unit circle
Hedenmalm, Korenblum, Zhu

Hardy space: background formalism

- Hardy space: analytic functions $g(z)$ in D :

$$\sup_{0 < r < 1} \int_0^{2\pi} \frac{d\theta}{2\pi} |g(re^{i\theta})|^p < \infty$$

here $p = 2$

- scalar product

$$(f, g) = \int_0^{2\pi} \frac{d\theta}{2\pi} [f(e^{i\theta})]^* g(e^{i\theta})$$

- orthonormal basis $\{z^N\}$, $N = 0, 1, 2, \dots$

$$f(z) = \sum_N f_N z^N$$

Z-transform (applications: signal processing)

- boundary function

$$f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta}) = \sum_{N=0}^{\infty} f_N e^{iN\theta}$$
$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} f(e^{i\theta}) = f_N; \quad N \geq 0$$
$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} f(e^{i\theta}) = 0; \quad N < 0$$

- phase states

$$\chi(z; \zeta) = \frac{(1 - |\zeta|^2)^{1/2}}{1 - \zeta z}; \quad |\zeta| < 1$$

$P(\theta) = |f(e^{i\theta})|^2$ phase distributions

phase states in the sense

$$P(\theta) = |\chi(e^{i\theta}; re^{i\phi})|^2 = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta + \phi)}$$

For $r \rightarrow 1$ we get $P(\theta) \rightarrow \delta(\theta + \phi)$.

- for any state $f(z)$

$$f(z) = (1 - |z|^2)^{-1/2} (\chi(\zeta; z^*), f(\zeta))$$

$\chi(z; w^*)$ orthogonal to $f(z) \leftrightarrow w$ zero of $f(z)$

Inner and outer states

$f(z)$ factorizes as:

$$f(z) = f_{\text{in}}(z)f_{\text{out}}(z)$$

- outer part of $f(z)$

$$f_{\text{out}}(z) = \exp[\Phi(z)]$$

where

$$\Phi(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} [2\mathcal{C}(r, \theta - \phi) - 1] \ln |f(e^{i\phi})|$$

and

$$\mathcal{C}(r, \theta) = (1 - re^{i\theta})^{-1}$$

the Cauchy kernel

phase distribution $P(\theta) = |f(e^{i\theta})|^2$ defines uniquely
outer part of $f(z)$.

outer part has no zeros

- inner part $f(z)$

$$f_{\text{in}}(z) = f_{\mathcal{Z}}(z) \exp[-\Phi(z)]$$

in the interior of the unit disk ($|z| < 1$)

$$|f_{\text{in}}(z)| \leq 1; \quad |f(z)| \leq |f_{\text{out}}(z)|$$

and on the unit circle ($|z| = 1$):

$$|f_{\text{in}}(e^{i\theta})| = 1; \quad |f(e^{i\theta})| = |f_{\text{out}}(e^{i\theta})| = [P(\theta)]^{1/2}$$

- Example of outer functions: phase states

$$\chi(z; \zeta) = \frac{(1 - |\zeta|^2)^{1/2}}{1 - \zeta z}; \quad |\zeta| < 1$$

- Examples of inner functions: most general bi-linear functions with $|f_{\text{in}}(e^{i\theta})| = 1$:

$$f(z) = \frac{\zeta - z}{1 - \zeta^* z}; \quad |\zeta| < 1$$

zero at $z = \zeta$

'building blocks' of more general inner functions.

- product of a finite number of such functions:

$$B(z; \{\zeta_N\}) = z^{p_0} \prod_{N=1}^K \left(\frac{\zeta_N - z}{1 - \zeta_N^* z} \right)^{p_N}$$

ζ_N : distinct numbers in the unit disk.

Zeros of $f(z)$: Blaschke products

- zeros ζ_N of a bounded analytic function in the unit disc satisfy:

$$\prod_{N=1}^{\infty} |\zeta_N|^{p_N} < \infty \leftrightarrow \sum_{N=1}^{\infty} p_N(1 - |\zeta_N|) < \infty \quad (1)$$

p_N multiplicity of zero ζ_N .

zeros move quickly towards the unit circle

sequence ζ_N in D which violates Eq.(1) \rightarrow corresponding phase states $\{\chi(z; \zeta_N)\}$ form an overcomplete set.

- sequence ζ_N in D which obeys Eq.(1)
inner function with those zeros: Blaschke product

$$B(z) = z^{p_0} \prod_{N=1}^{\infty} \left(\frac{\zeta_N^*}{|\zeta_N|} \frac{\zeta_N - z}{1 - \zeta_N^* z} \right)^{p_N}$$

not unique answer: $B(z) \exp[h(z)]$

- general function $f_Z(z)$ can be written as

$$f_Z(z) = f_{\text{in}}(z) f_{\text{out}}(z) = B(z) \exp[h(z)] f_{\text{out}}(z)$$

Discussion

- Analytic representation in the complex plane
growth/density of zeros
- Analytic representation in the unit disc
Bergmann space
Hardy space
- finite Hilbert spaces (eg spin)
complex functions $f(m)$ $m \in \mathbb{Z}_d$
stronger results when $m \in GF(p^\ell)$